GENERALIZED QUADRATURE FOR CALCULATING A TWO-DIMENSIONAL TEMPERATURE FIELD IN SEMIINFINITE BODIES WITH DISCONTINUOUS BOUNDARY CONDITIONS OF SECOND KIND

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A two-dimensional solution to the classical boundary-value problem of heat conduction is obtained in quadratures for a half-space. Heat is supplied here through a thin circular layer with the thermal flux density varying arbitrarily in time.

Two-dimensional transient temperature fields building up in semiinfinite bodies with local stationary and moving heat sources on their surface must be calculated in various areas of experimental physics, science, and engineering (examples being electron-beam and laser heating of massive objects, spot welding of metals, probing methods of nondestructive inspection for comprehensive determination of thermophysical characteristics of materials, etc.).

The unique correspondence between the one-dimensional transient temperature field $T(x, \tau)$ in a semiinfinite body and the thermal flux $q(\tau)$ impinging on its surface is in the theory of heat conduction described by the well-known quadrature expression [1]

$$T(x,\ \tau)-T_0=\frac{1}{b\sqrt{\pi}}\int\limits_0^{\tau}\exp\left[-\frac{x^2}{4a(\tau-\xi)}\right]\frac{q(\xi)}{\sqrt{\tau-\xi}}\,d\xi. \tag{1}$$

When the values of $T(x, \tau)$ are known and the values of $q(\tau)$ are not, then expression (1) constitutes an integral equation for determining the law according to which $q(\tau)$ varies (reverse problem of heat conduction).

The problem in this study will be to determine the two-dimensional transient temperature field $T(r, x, \tau)$ in cylindrical coordinates as a function of the thermal flux density $q(\tau)$ within a bounded circular region $0 \leqslant r < r_0$ on the surface of a semiinfinite body ($x = 0$). At the $x = 0$ surface we assume throughout the infinite region $\infty > r > r_0$ no temperature gradient in a direction normal to the boundary of the body. The initial temperature distribution is assumed to be uniform over all points of the semiinfinite body: $T_0 = \text{const}$. On this basis, the mathematically formulated problem reduces to the system of differential equations (with the origin of coordinates at the center of the heating spot of radius $r = r_0$ on surface $x = 0$):

for the region $0 \leqslant r < r_0$, $x \geqslant 0$ at time $\tau > 0$

$$\frac{\partial^2 T(r,\ x,\ \tau)}{\partial r^2}+\frac{1}{r}\frac{\partial T(r,\ x,\ \tau)}{\partial r}+\frac{\partial^2 T(r,\ x,\ \tau)}{\partial x^2}=\frac{1}{a}\frac{\partial T(r,\ x,\ \tau)}{\partial \tau}; \tag{2}$$

for the region $\infty > r > r_0$, $x \geqslant 0$ at time $\tau > 0$

$$\frac{\partial^2 \Theta(r,\ x,\ \tau)}{\partial r^2}+\frac{1}{r}\frac{\partial \Theta(r,\ x,\ \tau)}{\partial r}+\frac{\partial^2 \Theta(r,\ x,\ \tau)}{\partial x^2}=\frac{1}{a}\frac{\partial \Theta(r,\ x,\ \tau)}{\partial \tau}. \tag{3}$$

The initial condition for both equations will be stipulated as

$$T(r,\ x,\ 0)=\Theta(r,\ x,\ 0)=T_0=\text{const}, \tag{4}$$

and the boundary conditions will be

$$-\lambda\frac{\partial T(r,\ 0,\ \tau)}{\partial x}=q(\tau); \tag{5}$$

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$$\frac{\partial\Theta(r,\ 0,\ \tau)}{\partial x}=0; \tag{6}$$

$$\frac{\partial T(0,\ x,\ \tau)}{\partial r}=0 \ \text{(condition of symmetry)}; \tag{7}$$

$$\frac{\partial T(r,\ \infty,\ \tau)}{\partial x}=\frac{\partial\Theta(r,\ \infty,\ \tau)}{\partial x}=\frac{\partial\Theta(\infty,\ x,\ \tau)}{\partial r}=0; \tag{8}$$

$$T(r_0,\ x,\ \tau)=\Theta(r_0,\ x,\ \tau); \tag{9}$$

$$\frac{\partial T(r_0,\ x,\ \tau)}{\partial r}=\frac{\partial\Theta(r_0,\ x,\ \tau)}{\partial r}. \tag{10}$$

Applying to Eqs. (2) and (3) the infinite Fourier and Laplace integral transformations yields, for the given boundary conditions, the respective solutions in the form

$$T(r,\ x,\ \tau)-T_0=\frac{1}{b\sqrt{\pi}}\int\limits_0^\tau \exp\left[-\frac{x^2}{4a(\tau-\xi)}\right]\frac{q(\xi)}{\sqrt{\tau-\xi}}d\xi-$$

$$-\frac{r_0}{\lambda}\frac{1}{\pi^2 i}\int\limits_0^\infty\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\exp(s\tau)\frac{I_0\left(r\sqrt{p^2+\frac{s}{a}}\right)K_1\left(r_0\sqrt{p^2+\frac{s}{a}}\right)}{\sqrt{p^2+\frac{s}{a}}}\bar{q}(s)\cos pxdpds \tag{11}$$

for the region $0 > r > r_0$, $x \geqslant 0$ at time $\tau > 0$ and

$$\Theta(r,\ x,\ \tau)-T_0=\frac{r_0}{\lambda}\frac{1}{\pi^2 i}\int\limits_0^\infty\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\exp(s\tau)I_1\left(r_0\sqrt{p^2+\frac{s}{a}}\right)K_0\left(r\sqrt{p^2+\frac{s}{a}}\right)\frac{\bar{q}(s)\cos px}{\sqrt{p^2+\frac{s}{a}}}dpds \tag{12}$$

for the region $\infty > r > r_0$, $x \geqslant 0$ at time $\tau > 0$.

In expressions (11) and (12) p is the parameter of infinite Fourier cosine integral transformation

$$f_c(r,\ p,\ \tau)=\int\limits_0^\infty f(r,\ x,\ \tau)\cos pxdx; \tag{13}$$

$$\bar{q}(s)=\int\limits_0^\infty q(\tau)\exp(-s\tau)\,d\tau. \tag{14}$$

Let us consider the particular case of solution (11) on the axis at $x \geqslant 0$. Letting $r = 0$ (so that $I_0(0) = 1$) in expression (11), we integrate the latter with respect to parameter p and use for this the known value of the Sonin–Gegenbauer integral [2-5]. Then, upon applying the convolution theorem for the originals of two functions, we obtain the equation in quadratures

$$\Delta T(0,\ x,\ \tau)=\frac{1}{b\sqrt{\pi}}\int\limits_0^\tau \exp\left[-\frac{x^2}{4a(\tau-\xi)}\right]\left\{1-\exp\left[-\frac{r_0^2}{4a(\tau-\xi)}\right]\right\}\frac{q(\xi)\,d\xi}{\sqrt{\tau-\xi}} \tag{15}$$

for determining the excess temperature $\Delta T(0,\ x,\ \tau) = T(0,\ x,\ \tau) - T_0$ along $x \geq 0$ on the axis of the semiinfinite body. It is obvious that with $r_0 \to \infty$ we obtain the quadrature expression (1) for the one-dimensional case.

We will now use the quadrature expression (15) for determining the laws of temperature variation $\Delta T(0,\ x,\ \tau)$ in certain special cases of $q(\tau)$.

1. We assume that at time $\tau > 0$ on the surface $x = 0$ within the region $0 \leqslant r < r_0$ there appears a constant thermal flux of density $q(\tau) = q_0 = \text{const}$. According to the quadrature expression (15), the temperature field $\Delta T(0, x, \tau)$ along $x \geqslant 0$ on the axis of the semiinfinite body will then be

$$\Delta T(0,\ x,\ \tau) = \frac{q_0}{b\sqrt{\pi}} \int_0^{\tau} \exp\left[-\frac{x^2}{4a(\tau-\xi)}\right]\left\{1-\exp\left[-\frac{r_0^2}{4a(\tau-\xi)}\right]\right\} \times$$
$$\times \frac{d\xi}{\sqrt{\tau-\xi}} = \frac{2q_0\sqrt{\tau}}{b}\left[\operatorname{ierfc}\left(\frac{x}{2\sqrt{a\tau}}\right) - \operatorname{ierfc}\left(\frac{\sqrt{r_0^2+x^2}}{2\sqrt{a\tau}}\right)\right], \tag{16}$$

where

$$\operatorname{ierfc} z = \frac{1}{\sqrt{\pi}} \exp(-z^2) - z \operatorname{erfc} z.$$

Expression (16) for $\Delta T(0, x, \tau)$ is identical to the well-known solution [6].

2. We assume that at time $\tau > 0$ on the surface $x = 0$ within the region $0 \leqslant r < r_0$ there appears a thermal flux of a density which varies in time according to the law $q(\tau) = B/\sqrt{\tau}$, with $B = \text{const}$ characterizing a circular heat source. Then the quadrature expression (15) yields the equation

$$\Delta T(0,\ x,\ \tau) = \frac{B}{b\sqrt{\pi}} \int_0^{\tau} \exp\left[-\frac{x^2}{4a(\tau-\xi)}\right]\left\{1-\exp\left[-\frac{r_0^2}{4a(\tau-\xi)}\right]\right\} \times$$
$$\times \frac{d\xi}{\sqrt{\xi}\sqrt{\tau-\xi}} = \frac{B\sqrt{\pi}}{b}\left\{\operatorname{erfc}\left(\frac{x}{2\sqrt{a\tau}}\right) - \operatorname{erfc}\left(\frac{\sqrt{r_0^2+x^2}}{2\sqrt{a\tau}}\right)\right\} \tag{17}$$

for the temperature field $\Delta T(0, x, \tau)$ along $x \geqslant 0$ on the axis of the semiinfinite body.

The integration in Eq. (17) can be easily performed after prior change of variables to $t_1 = \frac{x^2}{4a(\tau-\xi)}$ and $t_2 = \frac{r_0^2+x^2}{4a(\tau-\xi)}$ and subsequent reduction of the integrals to the form found in tables

$$\int_0^{\infty} \frac{e^{-kt}dt}{\sqrt{t}(t+z)} = \frac{\pi}{\sqrt{z}} e^{kz} \operatorname{erfc} \sqrt{kz},$$
$$(\operatorname{Re} k > 0,\ z \neq 0,\ |\arg z| < \pi).$$

3. We assume that on the surface $x = 0$ within the region $0 \leqslant r < r_0$ there appears a thermal flux in the form of a rectangular pulse of duration τ_0:

$$q(\tau) = \begin{cases} q_0 = \text{const} & \text{at} \quad 0 < \tau < \tau_0, \\ 0 & \text{at} \quad \tau > \tau_0. \end{cases} \tag{18}$$

According to the quadrature expression (15), the excess temperature $\Delta T(0, x, \tau)$ along $x \geqslant 0$ on the axis of the semiinfinite body varies then according to the law

$$\Delta T(0,\ x,\ \tau) = \frac{q_0}{b\sqrt{\pi}} \int_0^{\tau_0} \exp\left[-\frac{x^2}{4a(\tau-\xi)}\right]\left\{1-\exp\left[-\frac{r_0^2}{4a(\tau-\xi)}\right]\right\} \frac{d\xi}{\sqrt{\tau-\xi}} = \frac{2q_0\sqrt{\tau}}{b}\left\{\operatorname{ierfc}\left(\frac{x}{2\sqrt{a\tau}}\right) -\right.$$
$$\left. - \operatorname{ierfc}\left(\frac{\sqrt{r_0^2+x^2}}{2\sqrt{a\tau}}\right) - U(\tau-\tau_0)\sqrt{\frac{\tau-\tau_0}{\tau}}\left[\operatorname{ierfc}\left(\frac{x}{2\sqrt{a(\tau-\tau_0)}}\right) - \operatorname{ierfc}\left(\frac{\sqrt{r_0^2+x^2}}{2\sqrt{a(\tau-\tau_0)}}\right)\right]\right\}. \tag{19}$$

For $\tau \leqslant \tau_0$ expression (19) becomes solution (16) for a constant thermal flux continuously present within the given region.

With x = 0 expression (19) yields the excess temperature as function of time $\Delta T(0, 0, \tau)$ at the point r = x = 0 located at the center of the heating spot on the surface of the semiinfinite body, namely

$$\Delta T(0,\ 0,\ \tau)=\frac{2q_0\sqrt{\tau}}{b\sqrt{\pi}}\left\{1-\sqrt{\pi}\ \mathrm{ierfc}\left(\frac{r_0}{2\sqrt{a\tau}}\right)-U(\tau-\tau_0)\sqrt{\frac{\tau-\tau_0}{\tau}}\left[1-\sqrt{\pi}\ \mathrm{ierfc}\left(\frac{r_0}{2\sqrt{a(\tau-\tau_0)}}\right)\right]\right\}. \quad (20)$$

4. We assume that at time $\tau > 0$ on the surface x = 0 within the region $0 \leqslant r < r_0$ there appears a thermal flux of a density

$$q(\tau)=q_0+k\tau, \quad (21)$$

which within the given region varies as a linear function of time, with k = const characterizing the heat source and q_0 = const.

Quadrature expression (15) for condition (21) yields the temperature field $\Delta T(0, x, \tau)$ along $x \geqslant 0$ on the axis of the semiinfinite body

$$\Delta T(0,\ x,\ \tau)=\frac{2\sqrt{\tau}}{b}\left[q_0+k\tau+\frac{1}{6}\frac{kx^2}{a}\right]\left\{\mathrm{ierfc}\left(\frac{x}{2\sqrt{a\tau}}\right)-\right.$$

$$\left.-\,\mathrm{ierfc}\left(\frac{\sqrt{r_0^2+x^2}}{2\sqrt{a\tau}}\right)\right\}-\frac{2k\tau\sqrt{\tau}}{3\sqrt{\pi}b}\exp\left(-\frac{x^2}{4a\tau}\right)\left\{1-\exp\left(-\frac{r_0^2}{4a\tau}\right)\right\}-\frac{kr_0^2\sqrt{\tau}}{3ba}\ \mathrm{ierfc}\left(\frac{\sqrt{r_0^2+x^2}}{2\sqrt{a\tau}}\right). \quad (22)$$

When k = 0, then expression (22) becomes solution (16) for $q(\tau) = q_0$ = const.

When q_0 = 0, then expression (22) readily yields the solution $\Delta T(0, x, \tau)$ for $q(\tau) = k\tau$ given. Letting x = 0 in expression (22), we obtain $\Delta T(0, 0, \tau)$ at the center point of the heating spot on the surface of the semiinfinite body for a thermal flux which within the spot region obeys law (21):

$$\Delta T(0,\ 0,\ \tau)=\frac{2\sqrt{\tau}}{b\sqrt{\pi}}(q_0+k\tau)-\frac{2\sqrt{\tau}}{b}\left[q_0+k\tau+\frac{kr_0^2}{6a}\right]\mathrm{ierfc}\left(\frac{r_0}{2\sqrt{a\tau}}\right)-\frac{2k\tau\sqrt{\tau}}{3\sqrt{\pi}b}\left[1-\exp\left(-\frac{r_0^2}{4a\tau}\right)\right], \quad (23)$$

considering that $\mathrm{ierfc}(0) = 1/\sqrt{\pi}$.

Letting x = 0 in the particular solutions (16) and (17), one can analogously establish the relation between temperature excess at the center of the heating spot (r = x = 0) and given flux density within the spot region on the body surface.

In the general case expression (15) for x = 0 becomes the particular quadrature expression which relates excess temperature $\Delta T(0, x, \tau) = T(0, 0, \tau) - T_0$ at the center of the heating spot (point r = x = 0) and density of the thermal flux crossing the surface x = 0 within the bounded circular region, namely

$$\Delta T(0,\ 0,\ \tau)=\frac{1}{b\sqrt{\pi}}\int_0^{\tau}\left\{1-\exp\left[-\frac{r_0^2}{4a(\tau-\xi)}\right]\right\}\frac{q(\xi)\,d\xi}{\sqrt{\tau-\xi}}. \quad (24)$$

When the quadrature expression (24) or the particular solutions obtained in this study for determining the relations $\Delta T(0, 0, \tau) = f[q(\tau), r_0, b, a, \tau]$ are used for analyzing the thermophysical characteristics of materials, then such equations can serve as the analytical basis for derivation of calculation formulas which will yield a, λ, b, and $c\gamma$ from measurements of temperature changes only at the center of the heating spot and thus for devising workable methods of nondestructive inspection (methods not involving distintegration of the test sample) such as the methods successfully used for determining the thermophysical properties of solids [7-10]. It is to be noted that all the complexity of practical implementation of such methods relates to the purely technical difficulties in producing specific controllable densities of thermal flux $q(\tau)$ entering through a specific region of the given semiinfinite body.

NOTATION

$T(r, x, \tau)$, temperature field in region $(0 \leqslant r < r_0, x \geqslant 0)$ of semiinfinite body at time $\tau > 0$; (r, x, τ), temperature field in region $\infty > r > r_0, x \geqslant 0)$ of semiinfinite body at time $\tau > 0$; r_0, radius of the heating spot; r, radial coordinate; τ, time; $q(\tau)$, thermal flux density inside the heating region (circular heat source) arbitrarily varying in time;

a, thermal diffusivity; λ, thermal conductivity; b, thermal activity of semiinfinite body; $\operatorname{erfc} X = \frac{2}{\sqrt{\pi}} \int_X^\infty e^{-t^2} dt$, probability integral; $\operatorname{ierfc} X = \int_X^\infty \operatorname{erfc} \xi d\xi$, multiple probability integral;

$\Delta T(r, x, \tau) = T(r, x, \tau) - T_0$, excess temperature of semiinfinite body in region $(0 \leqslant r < r_0, x \geqslant 0)$ at time $\tau > 0$; q_0, constant thermal flux density inside the heating spot; $B = q(\tau)\sqrt{\tau} =$ const, constant characterizing the variation of thermal properties of given circular local heat source as function of time; $\Delta T(0, x, \tau)$, excess temperature on the axis $(r = 0, x \geqslant 0)$ of semiinfinite body in selected system of coordinates at any instant of time $\tau > 0$; $\Delta T(0, 0, \tau)$, excess temperature; I_0 and I_1, modified Bessel functions of the first kind of respectively zeroth and first order; K_0 and K_1, modified Bessel functions of the second kind of respectively zeroth and first order; $U(\tau - \tau_0)$, symmetric unit-step function.

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